Real-space renormalization-group study of the phase transition in a Gaussian model of fractals

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In this paper the phase transition of the Gaussian model on m-sheet fractals $(m\mathrm{SG})_l$ and $(m\mathrm{DH})_l$ is investigated by the real-space renormalization-group method, i.e., decimation following a spin rescaling. The latter is introduced to keep the parameter b constant. Fixed points of the renormalization-group transformation are found and discussed. Our results show the existence of different properties of phase transition between the Gaussian model and the Ising model on fractals. In addition, we find that the critical point $k^*=b/4$ in a regular Sierpinski gasket is identified, with result of $k^*=b/d$ (d is the coordination number) in Euclidean space. This indicates that the critical point of the Gaussian model may be uniquely determined by the coordination number whether on homogeneous fractals or translationally invariant lattices. [S1063-651X(97)07506-5]

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I. INTRODUCTION

Ising and Ising-like models are the most widely discussed models on phase transitions and critical phenomena. During the course of recognition, many models were proposed for the purpose of a deeper understanding of properties and features of phase transitions. The Gaussian model is such a model which was proposed by Berlin and Kac [1] at first in order to make the Ising model more tractable. In the Gaussian model the spins are allowed to take any real value under a Gaussian-type distribution rather than the values ± 1 in the Ising model. On translationally invariant lattices the Gaussian model was exactly solvable. Later it was also investigated with mean-field theory and the momentum-space renormalization-group method [2,3]. Although an extension of the Ising model, the Gaussian model shows many differences from the Ising model in the properties of the phase transitions.

In the 1980s, Gefen *et al.* presented their series work [4–7] of the critical phenomena on fractals. Since then much attention has focused on the study of phase transitions of discrete spin models on fractals. However, to the authors' knowledge, no investigations have been found so far concerning the Gaussian model on fractal lattices.

Because fractals have dilation symmetry instead of translation symmetry, most powerful methods such as a Fourier transformation are unsuitable. In addition, the Monte Carlo method encountered trouble with infinite spin values in the Gaussian model. Fortunately, the real-space renormalization-group (RSRG) method has proved to be successful on this kind of self-similar lattice. In this paper we study the Gaussian model on some m-sheet Sierpinski gasket and a family of diamond-type hierarchical lattices (which can be treat as an m-sheet one-dimensional chain) by an exact decimation RSRG technique. In our scheme we have unusually combined decimation with a spin-rescaling procedure, which has never been applied to the usual decimation treatment, in order to get the proper renormalization-group (RG) recursion relation and critical point k^*. The paper is organized as follows. In Sec. II the m-sheet fractal is defined. In Sec. III the Gaussian model is discussed by the RSRG technique on a regular Sierpinski gasket as an example. In Sec. IV we give the results on some m-sheet fractals. Section V gives conclusions and a discussion.

II. *m*-SHEET FRACTALS

In this paper the RSRG technique is applied to study the phase transition of the Gaussian model to two families of m-sheet fractals: the m-sheet Sierpinski gasket (mSG) and diamond-type hierarchical (mDH) lattices. These two kinds of lattices can be attributed to the hierarchical lattice defined by Griffiths and Kaufman [8].

The m-sheet Sierpinski gasket, which is defined by Menezes and Magalhaes [9], is a generalization of the classical Sierpinski gasket (SG) [6,10–13]. The SG may be built iteratively from a generator $G(l)$: $G(l)$ is an equilateral triangle of side length l, which contains l layers of smaller unit side equilateral triangles among which only the $l(l+1)/2$ upward-oriented triangles survive, while the downward ones are empty. As Fig. 1(a) shows, the basic unit (the $n=0$ stage of the lattice) is an equilateral triangle. Replacing the basic unit with the generator $G(l)$, one gets the $n=1$ stage of the lattice. The $n+1$ stage of the structure of the lattice is obtained by replacing each smaller upward-oriented triangle of the stage n with the generator and the SG is obtained at the $n\to\infty$ limit. For the m-sheet SG, which is shown in Fig. 1(b), the basic unit is also an equilateral triangle. The generator $G(m,l)$ is generalized from $G(l)$. It is an m structure whose topology is similar to $G(l)$, with its $3m$ external sites (the vertices of a big triangle), connected m-fold times at three vertices (we call it roots hereafter). Then we can get the m-sheet SG with $G(m,l)$ as an iterative SG. The regular SG

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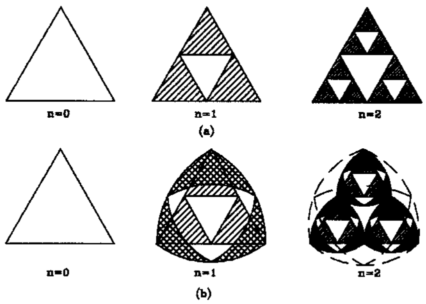

FIG. 1. First three stages of construction of $(m\mathrm{SG})_2$, with (a) $m=1$ and (b) $m=2$. The dashed lines in (b) indicate the same topological structure as the first sheet.

is a special case of the m-sheet SG [we call it $(m\mathrm{SG})_l$ later in this paper] with $m=1$ and $l=2$. The fractal dimension of $(m\mathrm{SG})_l$ is $D_f=\ln[ml(l+1)/2]/\ln l$.

The family of diamond-type hierarchical lattices is similar to the $(m\mathrm{SG})_l$. In fact, it is also a kind of m-sheet fractal, whose generators are composed of m sheets of one-dimensional chains with length l. So we call this family of lattices $(m\mathrm{DH})_l$, where the numbers of branches and bonds per branch are m and l, respectively. The construction of $(m\mathrm{DH})_l$ is shown in Fig. 2: The stage $n=0$ of the lattices is a line segment and the generator of $(m\mathrm{DH})_l$ is composed of m sheets of chains with length l, which are connected at two ends of the chain. It is obvious that $(1\mathrm{DH})_l$ corresponds to the one-dimensional chain and $(2\mathrm{DH})_2$ is the diamond-type hierarchical lattice. The fractal dimension is $D_f=\ln ml/\ln l=1+\ln m/\ln l$.

In the above two kinds of lattices, the ramification is finite when $m=1$ and infinite when $m>1$. For the Ising and Potts spin systems, it has been found that there is a finite-temperature phase transition of fractals with infinite ramification. It is naturally surmised that infinite ramification is a criterion of the existence of finite-temperature phase transitions. Much evidence shows that it may be true for discrete spin systems. In this paper we will show that this criterion cannot be extended to the Gaussian model.

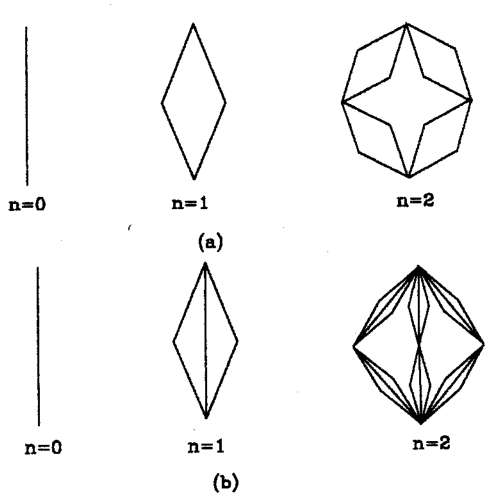

FIG. 2. First three stages of $(m\mathrm{DH})_2$, with (a) $m=2$ and (b) $m=3$.

There is another particular characteristic of the above m-sheet lattices: The number of neighbors is different at different sites. Some sites may have an infinite number of neighbors. This special characteristic prompts an investigation of the vibration spectrum of this kind of lattice [14,15]. We will show in this paper that this characteristic influences the temperature of the phase transition of the Gaussian model. In this paper, the case $l=2$ for the above two types of lattices is investigated.

III. DECIMATION RG OF THE GAUSSIAN MODEL ON THE SG

First of all, let us recall the Hamiltonian of the Ising model, which can be written as

$$-\beta\mathcal{H}=K\sum_{\{s\}} S_iS_j, \tag{1}$$

where $\beta=1/kT$, $K=J/kT$, J is the exchange integral, k is the Boltzmann constant, and T is the temperature. Each spin S_i can take only the discrete values of ± 1.

The Gaussian model is an extension of the Ising model in the following two aspects. First, the spin of the Gaussian model can take any real value between $-\infty$ and $+\infty$ in a single axis. Second, to prevent all spins from tending to infinity, a probability of finding a given spin between S_i and S_i+dS_i is assumed to be

$$\exp\left(-\frac{b}{2}S_i^2\right)dS_i, \tag{2}$$

which is a Gaussian-type distribution. In the expression b is a constant independent of temperature. Under this distribution the model stimulates the Ising model insofar as $\langle S_i\rangle=0$ and $\langle S_i^2\rangle=1$. The partition function of N particles is

$$Z_N=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}dS_1\cdots dS_N\,\exp\left(-\frac{b}{2}\sum_{i=1}^{N}S_i^2+K\sum_{\langle i,j\rangle}S_iS_j\right). \tag{3}$$

Here we assume $K>0$, which corresponds to the ferromagnetic interaction between neighboring spins. The $\langle i,j\rangle$ in the summation indicates that the sum should only count on the pair of nearest neighbors. From the expression of Z_N we can formally introduce an effective Hamiltonian $-\beta\mathcal{H}_{\mathrm{eff}}$, which is written as

$$-\beta\mathcal{H}_{\mathrm{eff}}=-\frac{b}{2}\sum_{i=1}^{N}S_i^2+K\sum_{\langle i,j\rangle}S_iS_j. \tag{4}$$

The first term in expression (4) can be interpreted as the self-energy of sites and the second is the interaction energy between neighboring spins.

Now let us apply the decimation RSRG technique to the Gaussian model. As usual, the decimation procedure is carried out by integrating the internal sites of each generator. This process corresponds to a scale transformation of length. Thus a new Hamiltonian with new self-energy and interaction parameters is obtained and the lattice returns to the previous stage of construction. Then the recursion relations that

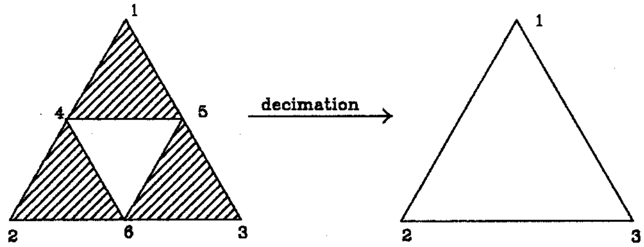

FIG. 3. Decimation procedure on a cell of the SG. The spins on sites 4–6 were integrated from $-\infty$ to ∞ in the procedure.

associated the new parameters with the old parameters can be derived by comparing the new Hamiltonian with the old.

For concreteness and explicitness we perform the above process (see Fig. 3) on $(1\mathrm{SG})_2$, i.e., the regular SG. The effective Hamiltonian is shown in expression (4). We consider the generator of the SG and define a restricted partition function of the generator

$$Z_{\text{cell}}=\int_{-\infty}^{\infty} dS_1\cdots dS_6\ \exp\left(K\sum S_iS_j-\frac{b}{2}\sum_{i=1}^{6} S_i^2\right). \quad (5)$$

After decimating the internal sites 4,5,6 by integrating spins S_4,S_5,S_6 from $-\infty$ to ∞, we obtain

$$Z_{\text{cell}}=A_0\int_{-\infty}^{\infty} dS_1dS_2dS_3\ \exp\left(K_a(S_1S_2+S_2S_3+S_3S_1)-\frac{b}{2}(1-2b_a)(S_1^2+S_2^2+S_3^2)\right), \quad (6)$$

where

$$K_a=\frac{K^2(b+2K)}{(b+K)(b-2K)},\quad b_a=\frac{bK^2}{(b+K)(b-2K)}. \quad (7)$$

The coefficient 2 in the term $2b_a$ comes from the contribution of two neighboring generators. Thus, after the decimation procedure, the partition function of the entire system is expressed as

$$Z=\int_{-\infty}^{\infty}[\Pi dS_m]\exp\left(-\frac{b}{2}(1-2b_a)\sum_i S_i^2+K_a\sum_{\langle i,j\rangle} S_iS_j\right) \quad (8)$$

and the corresponding Hamiltonian is

$$-\beta\mathcal{H}'=-\frac{b}{2}(1-2b_a)\sum_i S_i^2+K_a\sum_{\langle i,j\rangle} S_iS_j. \quad (9)$$

It should be noted that the coefficient of the first term in expression (9) is no longer independent of temperature. To make the coefficient independent of temperature and recover the original Gaussian distribution shown in expression (2), we carry out a rescaling treatment of spin as

$$S_i'=\sqrt{1-2b_a}S_i. \quad (10)$$

The partition function becomes

$$Z=\int_{-\infty}^{\infty}[\Pi dS_m']\exp\left(-\frac{b}{2}\sum_i S_i'^2+K'\sum_{\langle i,j\rangle} S_i'S_j'\right). \quad (11)$$

Thus the renormalized effective Hamiltonian is

$$-\beta\mathcal{H}_{\text{eff}}'=-\frac{b}{2}\sum_i S_i'^2+K'\sum_{\langle i,j\rangle} S_i'S_j', \quad (12)$$

where

$$K'=\frac{K_a}{1-2b_a}=\frac{K^2(b+2K)}{b^2-bK-6K^2} \quad (13)$$

is the recursion relation of the RG transformation. We can easily find that the fixed point of relation (13) is

$$K^*=\frac{b}{4}. \quad (14)$$

Then we can have $\lambda=(\partial k'/\partial k)_{k=k^*}=5>1$. This shows that this fixed point is unstable and thus is a critical point. From the familiar RG formula we can get the correlation exponent $\nu=\ln l/\ln\lambda=\ln 2/\ln 5$.

IV. RESULTS ON $(m\mathrm{SG})_2$ AND $(m\mathrm{DH})_2$

The above method can easily be extended to $(m\mathrm{SG})_2$. For the case of $(m\mathrm{SG})_2$, since the cell is an m sheet that connects only at the roots and each sheet of the generator gives the same contribution to the Hamiltonian after the scaling transformation of the length, it is easy to obtain

$$K_a=m\,\frac{K^2(b+2K)}{(b+K)(b-2K)},\quad b_a=m\,\frac{bK^2}{(b+K)(b-2K)}.$$

Similarly, after the rescaling transformation of spins $S_i'=\sqrt{1-2b_a}S_i$, we have

$$K'=\frac{K_a}{1-2b_a}=\frac{mK^2(b+2K)}{b^2-bK-(4m+2)K^2}. \quad (15)$$

This is the recursion relation of the RG transformation and the corresponding critical point is

$$K^*=\frac{\sqrt{m^2+26m+9}-(m+1)}{12m+4}\,b. \quad (16)$$

It can be checked that the fixed point is unstable for an arbitrary positive integer m. When $m\to\infty$, we have $K^*\to 0$ and $\lambda=(\partial k'/\partial k)_{k=k^*}\to 2$, where $\nu=1$.

The family of $(m\mathrm{DH})_2$ can be studied in the same way. The results are

$$K_a=m\,\frac{K^2}{b},\quad b_a=m\,\frac{K^2}{b^2},$$

$$K'=\frac{K_a}{1-2b_a}=\frac{mbK^2}{b^2-2mK^2} \quad (17)$$

and

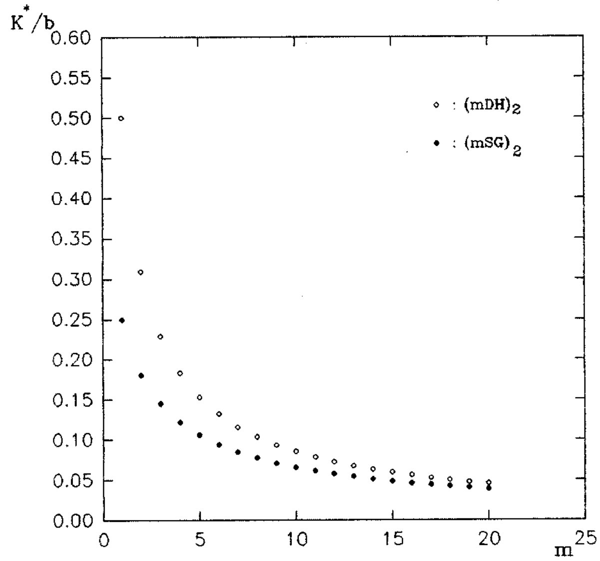

FIG. 4. Critical point of $(m\mathrm{SG})_2$ and $(m\mathrm{DH})_2$. b is a given constant that indicates the Gaussian-type distribution. When $m \to \infty$, $k^*/b \to 0$.

$$K^* = \frac{\sqrt{m^2+8m}-m}{4m}\, b. \tag{18}$$

In the special case of $m=1$, which is the one-dimensional chain, $K^*=b/2$, $\lambda=(\partial k'/\partial k)_{k=k^*}=4$, and the exponent $\nu=\frac{1}{2}$. These results completely agree with the previous results in Euclidean space [2]. When $m\to\infty$, we have $K^*\to 0$, $\lambda=(\partial k'/\partial k)_{k=k^*}\to 2$, and $\nu=1$, as for $(m\mathrm{SG})_2$. The relations of K^* vs m and ν vs m on these two families of fractals are shown in Figs. 4 and 5.

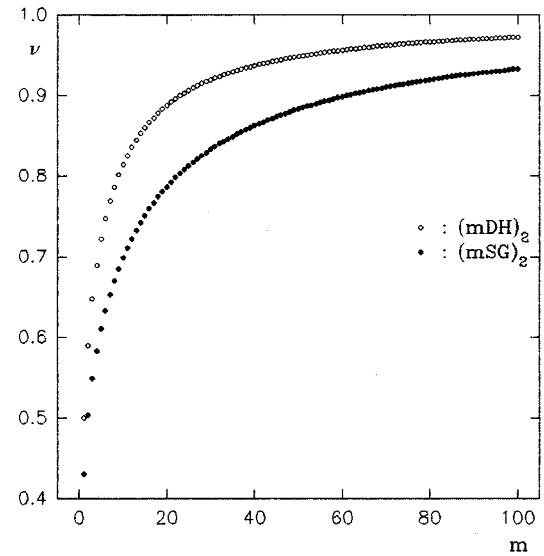

FIG. 5. Critical exponent ν of $(m\mathrm{SG})_2$ and $(m\mathrm{DH})_2$. When $m\to\infty$, $\nu\to 1$.

V. CONCLUSIONS AND DISCUSSION

In this paper we have applied the real-space renormalization-group method to the study of the phase transition of the Gaussian model on two families of m-sheet fractals $(m\mathrm{SG})_l$ and $(m\mathrm{DH})_l$ for the case $l=2$. Recursion relations of the renormalization group and fixed points (critical point) are found. In addition to the scale transformation of length, a rescaling of spins is adopted in our scheme in order to keep the Gaussian distribution parameter b constant during the renormalization-group procedure. This spin rescaling is beyond the regular decimation RG method; it has proved to be successful on the Gaussian model in this paper.

In the two families of lattices mentioned, we can find an unstable fixed point in the finite-temperature zone even when $m=1$. This means that the finite-temperature phase transition occurs in the Gaussian model on finite-ramification lattices. It is very different from the Ising model, which exhibits only a finite-temperature phase transition on infinite-ramification lattices. Thus the criterion for the phase transition mentioned in Sec. II is no longer suitable for the Gaussian model, though the spin values in the Gaussian model are restricted to a single axis.

We have also noticed that the critical point that we found in the regular SG fractal is $k^*=b/4$, where 4 is the coordination number of the SG. It is identified with the result $K^*=b/d$ in Euclidean space, where d is the coordination number. This identity seems to indicate that for a homogeneous lattice (whose coordinate number is equal everywhere), whether the critical point of the Gaussian model is translational symmetric or dilational symmetric, it is uniquely determined by the coordination number. This conjecture seems reasonable, but still needs confirmation.

In the cases $m>1$ we discussed, since the coordinate number is not constant, the critical point is dependent of geometrical details of the lattice. It is also an interesting problem for further investigations.

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